Classical descriptive set theory, generalized descriptive set theory, and $\text{I}_0$

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G. Cantor (1870) proved that all closed subsets of $\mathbb{R}$ have the PSP: this is one of the earlier results in the area now called (classical) descriptive set theory.
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to ensure that e.g. both \( \kappa^2 \) and \( \kappa \kappa \) have a separability-like condition (i.e. they have a dense subset of size \( \kappa \)).
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The resulting theory is extremely rich and interesting, but quite different from the classical one: most of the nontrivial results are either simply false or at least independent of ZFC when $\kappa > \omega$ (e.g. both the Lusin’s separation theorem and Souslin’s theorem fail).
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More recently, Woodin suggested to study generalized DST under $I_0$ in connection with his study of the model $L(V_{\lambda+1})$ (where $\lambda$ is the witness of $I_0$). Notice that such a $\lambda$ has always countable cofinality.
The axiom I0

$I_0(\lambda)$ is the statement: There is a nontrivial elementary embedding $j: L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$ (we call $j$ a witness to $I_0(\lambda)$).
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$$O_{a,\alpha} = \{ X \in V_{\lambda+1} \mid X \cap V_\alpha = a \}$$

for $\alpha < \lambda$ and $a \subseteq V_\alpha$. 
The axiom 10

10(\lambda) is the statement: There is a nontrivial elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$ (we call $j$ a witness to $10(\lambda)$).

10 is the statement: there is $\lambda$ for which $10(\lambda)$.

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Woodin claims that “the theory of $\mathcal{P}(V_{\lambda+1})$ in $L(V_{\lambda+1})$ under 10(\lambda) is reminiscent of the theory of $\mathcal{P}(\mathbb{R})$ in $L(\mathbb{R}) = L(V_{\omega+1})$ under AD”.
A test for Woodin’s claim is the Perfect Set Property PSP.
I0 and Woodin’s analysis

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**Theorem (Woodin)**

Assume $I_0(\lambda)$, as witnessed by $j$. Every $U(j)$-representable set $A \subseteq V_{\lambda+1}$ in $L(V_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $\omega^2$ topologically embeds into $A$. 

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A test for Woodin’s claim is the Perfect Set Property PSP. Some of the following statements involve $U(j)$-representability, which is a technical notion isolated by Woodin reminiscent of the one of $\kappa$-weakly homogenously Souslin sets.

**Theorem (Woodin)**

Assume $\text{I}_0(\lambda)$, as witnessed by $j$. Every $U(j)$-representable set $A \subseteq V_{\lambda+1}$ in $L(V_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $\omega^2$ topologically embeds into $A$.

**Theorem (Shi)**

Assume $\text{I}_0(\lambda)$, as witnessed by $j$. Then every set $A$ in $L_\lambda(V_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $C(\lambda) = \prod_{i \in \omega} \lambda_i$ topologically embeds into $A$, where $\lambda_i \uparrow \lambda$. 
I0 and Woodin’s analysis

Theorem (Shi)

Assume I0(\(\lambda\)), as witnessed by \(j\). Assume that all subsets of \(V_{\lambda+1}\) in \(L(V_{\lambda+1})\) are \(U(j)\)-representable.

Theorem (Cramer)

Assume I0(\(\lambda\)), as witnessed by \(j\). Every \(A \subseteq V_{\lambda+1}\) in \(L(V_{\lambda+1})\) satisfies the following dichotomy: either \(|A| \leq \lambda\) or \(B(\lambda) = \omega\) topologically embeds into \(A\).

Except for Woodin's result (which is the weakest!), the proofs are quite different from the classical ones dealing with the (classical) PSP, and indeed technical machineries specific to the model \(L(V_{\lambda+1})\) under I0(\(\lambda\)) are heavily involved.
Theorem (Shi)

Assume \( \text{I}0(\lambda) \), as witnessed by \( j \). Assume that all subsets of \( V_{\lambda+1} \) in \( L(V_{\lambda+1}) \) are \( U(j) \)-representable. Then every \( A \subseteq V_{\lambda+1} \) in \( L(V_{\lambda+1}) \) satisfies the following dichotomy: either \( |A| \leq \lambda \) or \( C(\lambda) = \prod_{i \in \omega} \lambda_i \) topologically embeds into \( A \), where \( \lambda_i \nearrow \lambda \).

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Our goal is to study the generalized Cantor space $\lambda^2$ when $\lambda$ is singular. We denote by $\lambda_i$ any sequence of length $\mu = \text{cof}(\lambda)$ cofinal in $\lambda$. 
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**Proposition (Džamonja-Väänänenen, Dimonte-M.)**

The following spaces are homeomorphic (products of length $\mu$ are endowed with the $< \mu$-supported product topology):

- $\lambda^2$
- $\prod_{i<\mu} \lambda_i$, where each $\lambda_i$ is discrete;
- $\mu^{(2<\lambda)}$, where $2^{<\lambda}$ is discrete.
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- $\mu(2^{<\lambda})$, where $2^{<\lambda}$ is discrete.
Dropping the first half of the usual condition

\[ \lambda^{<\lambda} = \lambda \quad \equiv \quad \text{cof}(\lambda) = \lambda \quad \text{and} \quad 2^{<\lambda} = \lambda \quad (\dagger) \]

we remain with a singular \( \lambda \) satisfying \( 2^{<\lambda} = \lambda \).
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we remain with a singular $\lambda$ satisfying $2^{<\lambda} = \lambda$ or, equivalently, with a singular strong limit $\lambda$. 

These very simple observations have lot consequences.
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$$\lambda^2 \approx \prod_{i<\mu} \lambda_i \approx \mu \lambda.$$
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Moreover, in this case $\lambda^2 \not\approx \lambda^\lambda$ because the latter has density $\lambda^{<\lambda} > \lambda$.
(Indeed, $\lambda^2$ and $\lambda^\lambda$ may even fail to be $(\lambda^+\text{-})\text{Borel isomorphic.}$)
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If furthermore $\text{cof}(\lambda) = \omega$, then we get

$$\lambda^2 \approx C(\lambda) \approx B(\lambda).$$
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If furthermore $\text{cof}(\lambda) = \omega$, then we get

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Thus when $\lambda$ is strong limit of countable cofinality, the generalized Cantor space $\lambda^2$ is a completely metrizable space of density $\lambda$, briefly: a $\lambda$-Polish space.
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A metric space $X$ is said **uniformly zero-dimensional** if for every $\varepsilon > 0$, every open set of $X$ can be partitioned into *clopen* sets with diameter $< \varepsilon$.

(Uniform zero-dimensionality follows from ultrametrizability and is equivalent to ultraparacompactness.)
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**Proposition (Dimonte-M.)**

Let $\lambda > \omega$ be strong limit of countable cofinality.

- $\lambda^2$ is universal for uniformly zero-dimensional $\lambda$-Polish spaces.
  
  (A space $X$ is a uniformly zero-dimensional $\lambda$-Polish space iff it is homeomorphic to a closed subset of $\lambda^2$, iff it admits a compatible complete ultrametric).
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- Every closed subset $C$ of a uniformly zero-dimensional $\lambda$-Polish space $X$ is a retract of it.
  
  *(There is a continuous surjection $g : X \to C$ with $g \upharpoonright C = \text{id}_C$.)*
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- Every closed subset $C$ of a uniformly zero-dimensional $\lambda$-Polish space $X$ is a retract of it. (There is a continuous surjection $g: X \to C$ with $g \upharpoonright C = \text{id}_C$.)
- Every nonempty $\lambda$-Polish space is a continuous image of $\lambda^2$. 
Woodin’s approach to the study of $V_{\lambda+1}$ falls in this setup as well. Recall that $V_{\lambda+1}$ is endowed with the topology generated by $O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap V_\alpha = a\}$ for $\alpha < \lambda$ and $a \subseteq V_\alpha$. 
The generalized Cantor space $\lambda^2$ and Woodin’s $L(V_{\lambda+1})$

Woodin’s approach to the study of $V_{\lambda+1}$ falls in this setup as well. Recall that $V_{\lambda+1}$ is endowed with the topology generated by $O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap V_\alpha = a\}$ for $\alpha < \lambda$ and $a \subseteq V_\alpha$.

**Lemma**

If $\text{cof}(\lambda) = \omega$ and $\lambda_i \nearrow \lambda$, then

$$V_{\lambda+1} \approx \prod_{i \in \omega} |V_{\lambda_i+1}| \approx \omega \left( \sup_{i \in \omega} \beth_{\lambda_i+1} \right) \approx \omega \left( \beth_\lambda \right).$$
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**Lemma**

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If furthermore $\lambda$ is limit of inaccessible cardinals (which is the case under $I_0(\lambda)$), then

$$V_{\lambda+1} \approx \prod_{i \in \omega} \lambda_i \approx \omega \lambda \approx \lambda^2.$$
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Notice also that if $\lambda$ is singular then

$$\lambda^+\text{-Borel} = \lambda\text{-Borel}.$$
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Similar results hold for the generalized Baire space $\lambda\lambda$. 
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3. \( A \) is a continuous image of a closed \( F \subseteq \omega \omega \)
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There are some problems when trying to generalize these equivalences by replacing \( \omega^2 \) and \( \omega \omega \) with \( \kappa^2 \) and \( \kappa^\kappa \), especially when \( \kappa \) is regular.

However...
If $\text{cof}(\lambda) = \omega$ and $\lambda$ is strong limit, TFAE:

1. $A$ is a continuous image of a $\lambda$-Polish space
2. $A = \emptyset$ or $A$ is a continuous image of $\omega$
3. $A$ is a continuous image of a closed $F \subseteq \omega^\lambda$
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5. $A$ is the projection of a closed subset of $X \times \omega^\lambda$
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This is exactly the notion of a $\lambda$-analytic set isolated by Stone.

Remark: One may be tempted to generalize the notion of "analytic" as "continuous image of a closed subset of $\lambda^\lambda$", as in the regular case. However, this would give a much coarser definition, encompassing $\lambda$-analytic sets, $\lambda$-coanalytic sets, $\Sigma_1^2(\lambda)$ sets, and, under the assumption that $\lambda < \lambda$ is large, also all $\lambda$-projective sets.
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λ-Analytic sets

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Assume again that \( \lambda \) is strong limit with countable cofinality.
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**Proposition (Dimonte-M.)**

The collection of all $\lambda$-analytic sets (properly) contain the $\lambda^{(+)}$-Borel ones.
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**Generalized Lusin’s separation theorem (Dimonte-M.)**

If $A, B$ are disjoint analytic subsets of a $\lambda$-Polish space, then $A$ can be separated from $B$ by a $\lambda$-Borel set.
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This has many consequences:
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This has many consequences:

- a function is \( \lambda \)-Borel iff its graph is \( \lambda \)-analytic, iff its graph is \( \lambda \)-Borel;
λ-analytic vs λ-Borel

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The collection of all λ-analytic sets (properly) contain the λ^(+)-Borel ones.

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This has many consequences:

- a function is λ-Borel iff its graph is λ-analytic, iff its graph is λ-Borel;
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Assume again that $\lambda$ is strong limit with countable cofinality.

**Proposition (Dimonte-M.)**

The collection of all $\lambda$-analytic sets (properly) contain the $\lambda^{(+)}$-Borel ones.

**Generalized Lusin’s separation theorem (Dimonte-M.)**

If $A, B$ are disjoint analytic subsets of a $\lambda$-Polish space, then $A$ can be separated from $B$ by a $\lambda$-Borel set.

**Generalized Souslin’s theorem (Dimonte-M.)**

A subsets of a $\lambda$-Polish space is $\lambda$-bianalytic iff it is $\lambda^{(+)}$-Borel.

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Definition

A subset $A$ of a topological space $X$ has the $\lambda$-PSP if either $|A| \leq \lambda$, or else $\lambda^2$ topologically embeds into $A$. 

Similarly to the classical case

Theorem (essentially A. H. Stone)

Let $\lambda$ be strong limit of countable cofinality. Every $\lambda$-analytic subset of a uniformly zero-dimensional $\lambda$-Polish space has the $\lambda$-PSP.

What for more complicated sets?

Motivated by the fact that, in the classical context, $\kappa$-homogeneously Souslin sets have the PSP (and inspired by Woodin's notion of $U(j)$-representability), we developed the following machinery.

L. Motto Ros (Turin, Italy)
### Definition

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(\mathbb{U}, \kappa)-representable sets

Definition

A family \( \mathbb{U} \) of ultrafilters is **orderly** iff there exists a set \( K \) such that for all \( \mathcal{U} \in \mathbb{U} \) there is \( n \in \omega \) for which \( {}^nK \in \mathcal{U} \). Such an \( n \) is called the **level** of \( \mathcal{U} \).
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A **tower** of ultrafilters in such a \( \mathbb{U} \) is a sequence \( (\mathcal{U}_i)_{i \in \omega} \) such that for all \( m < n < \omega \):

- \( \mathcal{U}_n \in \mathbb{U} \) has level \( n \);
- \( \mathcal{U}_n \) projects to \( \mathcal{U}_m \), i.e. for each \( A \subseteq mK \) we have
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**L. Motto Ros (Turin, Italy)**
From now on $\lambda$ is strong limit with $\text{cof}(\lambda) = \omega$, and $\lambda_i \uparrow \lambda$. 

Remark 1: If $\lambda = \omega$ and $A \subseteq \omega^\omega$ is $\kappa$-weakly homogenously Souslin, then $A$ is $(U, \kappa)$-representable for a suitable orderly family of ultrafilters $U$. 

Remark 2: Exploiting the natural homeomorphism between $V_{\lambda + 1}$ and $\omega^\lambda$ the above definition yields Woodin's $U(j)$-representability when $\kappa = \lambda^+$ and $U$ is a certain family of ultrafilters usually denoted by $U(j, \kappa, (\alpha_i)_{i \in \omega})$. 

$(U, \kappa)$-representable sets
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**Definition**

Let \( \kappa \geq \lambda \) be a cardinal, and let \( \mathbb{U} \) be an orderly family of \( \kappa \)-complete ultrafilters. A \((\mathbb{U}, \kappa)\)-representation for \( Z \subseteq \omega \lambda \) is a function \( \pi: \bigcup_{i \in \omega} i \lambda \times i \lambda \rightarrow \mathbb{U} \) such that:

1. If \( s, t \in i \lambda \), then \( \pi(s, t) \) has level \( i \).
2. For any \( (s, t) \in n \lambda \) if \( (s', t') \sqsubseteq (s, t) \) then \( \pi(s', t') \) projects to \( \pi(s, t) \).
3. \( x \in Z \) iff there is \( y \in \omega \lambda \) s.t. \( (\pi(x \upharpoonright i, y \upharpoonright i)) i \in \omega \) is well-founded.

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The following condition turns out to be very helpful when checking well-foundness of towers of ultrafilters.

**Definition**

A \((U, \kappa)\)-representation \(\pi\) for a set \(Z \subseteq \omega \lambda\) has the **tower condition** if there exists \(F: \text{ran}\(\pi\) \to \bigcup U\) such that:

- \(F(U) \in U\) for all \(U \in \text{ran}(\pi)\);
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**Remark 2:** Woodin has an analogous notion of "tower condition" in the context of \(U(j)\)-representability.

Cramer later proved that if \(I_0(\lambda)\) holds, then all \(U(j)\)-representable sets in \(P(V_\lambda + 1) \cap L(V_\lambda + 1)\) admit in fact a \(U(j)\)-representation with the tower condition.

L. Motto Ros (Turin, Italy)
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Let $\lambda$ be strong limit with $\text{cof}(\lambda) = \omega$, and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq \omega^\lambda$ admits a $(U, \kappa)$-representation with the tower condition, then $Z$ has the $\lambda$-PSP.
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**Corollary**

Assume $I_0(\lambda)$, as witnessed by $j$. If $A \in \mathcal{P}(V_{\lambda+1}) \cap L(V_{\lambda+1})$ is $U(j)$-representable, then $A$ has the $\lambda$-PSP.
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**Corollary**

Assume $I^0(\lambda)$, as witnessed by $j$. If $A \in \mathcal{P}(V_{\lambda+1}) \cap L(V_{\lambda+1})$ is $U(j)$-representable, then $A$ has the $\lambda$-PSP.

**Corollary**

Assume $I^0(\lambda)$. All $\lambda$-projective subsets of any uniformly zero-dimensional $\lambda$-Polish space have the $\lambda$-PSP.
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**Corollary**

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**Proof.**
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**Proof.** Let $\pi$ be a $(U, \kappa)$-representation for $Z$ with the tower condition, as witnessed by $F$. 
Proof of the main theorem

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Let \( \lambda \) be strong limit with \( \text{cof}(\lambda) = \omega \), and let \( \kappa \geq \lambda \) be a cardinal. If \( Z \subseteq \omega \lambda \) admits a \((U, \kappa)\)-representation with the tower condition, then \( Z \) has the \( \lambda \)-PSP.

**Proof.** Let \( \pi \) be a \((U, \kappa)\)-representation for \( Z \) with the tower condition, as witnessed by \( F \). Let \( G(Z) \) (or rather \( G(\pi, F) \)) be the game

\[
\begin{array}{c|c|c|c|c}
I & & & & \\
\hline
II & & & & \\
\end{array}
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<table>
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<tr>
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<th>$(s^0_i, t^0_i)_{i&lt;\lambda_0}$</th>
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- $s^k_i, t^k_i \in j_k \mu_k$ for some $\mu_k < \lambda$ and $j_k \in \omega$, with $s^k_i \neq s^k_{i'}$ if $i \neq i'$;
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Let $\lambda$ be strong limit with $\text{cof}(\lambda) = \omega$, and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq \omega \lambda$ admits a $(\mathbb{U}, \kappa)$-representation with the tower condition, then $Z$ has the $\lambda$-PSP.

**Proof.** Let $\pi$ be a $(\mathbb{U}, \kappa)$-representation for $Z$ with the tower condition, as witnessed by $F$. Let $G(Z)$ (or rather $G(\pi, F)$) be the game

\[
\begin{array}{c|cc}
I & (s^0_i, t^0_i)_{i<\lambda_0} & \\
II & i_0 & \\
\end{array}
\]

- $s^k_i, t^k_i \in j_k \mu_k$ for some $\mu_k < \lambda$ and $j_k \in \omega$, with $s^k_i \neq s^k_{i'}$ if $i \neq i'$;
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$I$ \quad $\begin{array}{c|c|c}
(s_i^0, t_i^0)_{i<\lambda^0} & z_0, \\
i_0
\end{array}$

$\begin{array}{c|c|c}
\hline
\text{II} & \text{I win if she can play for infinitely many turns.}
\end{array}$

- $s_i^k, t_i^k \in j_k \mu_k$ for some $\mu_k < \lambda$ and $j_k \in \omega$, with $s_i^k \neq s_i^{k'}$ if $i \neq i'$;
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L. Motto Ros (Turin, Italy)
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When I wins a run, she has built an element $x = \bigcup_{k \in \omega} s^k_{i_k} \in \omega \lambda$, and a
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Consider the auxiliary game \( G^*(Z) \) (or rather \( G(\pi) \))
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When I wins a run, she has built an element \( x = \bigcup_{k \in \omega} s_{i_k}^k \subseteq \omega \lambda \), and a \( y = \bigcup_{k \in \omega} t_{i_k}^k \subseteq \omega \lambda \) witnessing that \( x \in Z \) — the well-foundedness of the corresponding tower is witnessed by \( z = \bigcup_{k \in \omega} z_k \), since \( z_k \in F(\pi(s_{i_k}^k, t_{i_k}^k)) \).

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\[
\begin{array}{c|c|c}
I & (s_i^0, t_i^0)_{i < \lambda_0} & \rule{0pt}{2.5ex} \\
\hline
II & (s_{i_0}^0, t_{i_0}^0)_{i_0 < \lambda_0} & \rule{0pt}{2.5ex}
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Consider the auxiliary game $G^*(Z)$ (or rather $G(\pi)$)

\[
\begin{array}{c|c|c}
I & (s_i^0, t_i^0)_{i<\lambda_0} & (s_i^1, t_i^1)_{i<\lambda_1} \\
\hline
\Pi & i_0 & i_0 \\
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where I does not have to produce the witnesses $z_k$,
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Consider the auxiliary game \( G^*(Z) \) (or rather \( G(\pi) \))

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Proof of the main theorem

When I wins a run, she has built an element \( x = \bigcup_{k \in \omega} s^k_{i_k} \in \omega \lambda \), and a
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| I        | \((s^0_i, t^0_i)_{i<\lambda_0}\) | \((s^1_i, t^1_i)_{i<\lambda_1}\) | \((s^2_i, t^2_i)_{i<\lambda_2}\) | \(\ldots\)
|----------|---------------------------------|---------------------------------|---------------------------------|---------|
| II       | \(i_0\)                         | \(i_1\)                         | \(i_2\)                         | \(\ldots\)

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\hline
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where I does not have to produce the witnesses \( z_k \), and I wins iff
\( x = \bigcup_{k \in \omega} s^k_{i_k} \in Z \) with \( y = t^k_{i_k} \) witnessing this. A priori, such a game is
not necessarily determined (the complexity of the payoff depends on the
complexity of \( Z \) and \( \pi \)), but...
Proof of the main theorem

...any winning strategy \( \tau \) of II in \( G(Z) \) can be converted into a winning strategy \( \tau^* \) of II in \( G^*(Z) \).
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**Claim.** If II wins $G^*(Z)$, then $|Z| \leq \lambda$. 
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Claim. If II wins \( G^*(Z) \), then \( |Z| \leq \lambda \).

Given a position \( p \) in the game \( G^*(Z) \) consisting of \( k \)-many rounds, let \( A_p \) be the set of those \( s_{i_k}^{k-1} \sqsubseteq x \in \omega \lambda \) for which whatever I plays in her next turn, the answer by II following \( \tau^* \) is such that \( s_{i_k}^k \nsubseteq x \).
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L. Motto Ros (Turin, Italy)
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Thank you for your attention!